APRIL WRAP-UP

Icosatwins (P418). This reverse-video version of Problem 1099 in *The College Mathematics Journal* came to us from Wenwen Du and Paul Peck of Glenville State College. You are going to color the triangular faces of an icosahedron blue and white so that every white triangle shares exactly one of its edges with another white triangle. What is the maximum number of white triangles that your icosahedron can have?

There can be at most **12** triangles colored white. The Missouri State problem-solving group submitted this solution, and the Georgia Southern University solvers sent a different proof.

Consider the five triangles surrounding any vertex. If four or more are white, then at least one white triangle will have two neighboring triangles shaded in white. Hence, there are at most three white triangles meeting at any vertex. But there are 12 vertices in the icosahedron, and each triangle touches three vertices, so there are at most $12 \times 3 / 3 = 12$ white triangles, as desired. Figure 2 shows a coloring that achieves this upper bound (note the exterior unbounded region represents the twelfth white triangle).

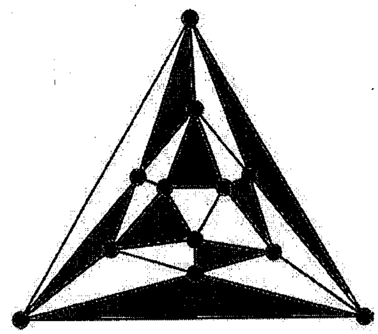

Figure 2. Face-coloring of an icosahedron with 12 white triangles, each adjacent to exactly one other.

The Missouri State group also notes that the centers of the blue triangles of the coloring in figure 2 comprise the vertices of a cube (of edge length $\phi^2 / 3$). Moreover, the centroids of the six white regions form an octahedron of *unit* edge length—if you find a compact proof of this fact, please submit it to the Playground.

SET Another Problem (P420). This problem came from Hsu, Ostroff, and Van Meter's article "Set with a Twist." Given a group G with identity e, a 3SET in G is an ordered triple $\langle g_1, g_2, g_3 \rangle$ of three distinct elements of G such that $g_1 g_2 g_3 = e$.

How many 3SETs are there in a) S_4, b) S_5, or c) S_6? Here S_n is the usual symmetric group, or group of permutations, on n elements.

Dmitry Fleischman and the Missouri State solvers sent solutions, along with partial answers from Wayne Nelson (Lovelock Correctional) and the Georgia Southern solvers. There are **522**, **14082**, and **516402** 3SETs in S_4, S_5, and S_6, respectively.

Note that any ordered pair $\langle g, h \rangle$ from G participates in at most one 3SET, namely $\langle g, h, (gh)^{-1} \rangle$. So when $|G| = N$, we begin with the N^2 ordered pairs and count 3SETs by inclusion-exclusion.

There are three ways this triple could fail to be a 3SET: $g = h$, $g = (gh)^{-1}$, or $h = (gh)^{-1}$. There are N pairs $\langle g, h \rangle$ that satisfy any one of these three equations. Any two or all three of them holding simultaneously is equivalent to $g^3 = e$. Hence, if we let N_3 be the number of elements whose cube is the identity, the number of 3SETs is

$$N^2 - 3N + 3N_3 - N_3 = N(N-3) + 2N_3.$$

For $G = S_n$, $N = n!$. For $n < 6$, the only elements of order 1 or 3 are the identity and the three cycles, yielding $1 + n(n-1)(n-2)/3$ such elements. However, S_6 contains additional elements of order 3: the products of two disjoint three-cycles, of which there are (in S_n) $n!/((n-6)! \cdot 3^2 \cdot 2)$. The numerical values above follow.